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BANDED PERTURBATIONS OF THE UNIT VECTOR BASIS IN SOME SEQUENCE SPACES

BY

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ABSTRACT

Perturbations of the unit vector basis of the form $x_n = \sum_{|j-n|\leqq m} a_{nj}e_j$ where m is a fixed positive integer are investigated. It is shown that if $|a_{nj}| \leqq 1$ and if $\{x_n\}$ possesses a biorthogonal sequence uniformly bounded in l_p for some $1 \leqq p < \infty$, then $\{x_n\}$ is equivalent to $\{e_n\}$ in every interpolation space of l_1 and l_∞. In particular, if $\{x_n\}$ is a seminormalized basic sequence in some reflexive Orlicz space l_N, then $\{x_n\}$ is equivalent to $\{e_n\}$ in l_N.

Szankowski and Zippin have shown that if $x_n = a_ne_n - b_ne_{n+1}$ is a seminormalized sequence in l_p having a seminormalized biorthogonal sequence $\{f_n\}$ in l_p^*, then $\{x_n\}$ is a basic sequence equivalent to $\{e_n\}$, the unit vector basis. We show that a similar result is true for perturbations of the form $x_n = \sum_{|j-n|\leqq m} a_{nj}e_j$ where m is a fixed positive integer. Essential use is made of the result of [2] that the elements of the inverse of a banded matrix decay uniformly exponentially to zero as they move away from the diagonal. An intermediate step shows that such perturbations are bases if and only if they possess uniformly bounded biorthogonal linear functionals.

With $\{e_n\}$ the unit vector basis, we call $\{x_n = \sum_{|j-n|\leqq m} a_{nj}e_j\}$ a *normalized banded perturbation* of $\{e_n\}$ if $|a_{nj}| \leqq 1$ for all n, j. A Banach space X is called an *interpolation space* of l_1 and l_∞ if every linear operator bounded on both l_1 and l_∞ is also bounded on X. A theorem of Calderón [1, theor. 3] implies that all Orlicz sequençe spaces, cf. [3], are interpolation spaces of l_1 and l_∞. Note that if $\{x_n\}$ is a normalized banded perturbation of $\{e_n\}$ with $x_n = \sum_{|j-n|\leqq m} a_{nj}e_j$ and if $A = (a_{ij})$, then A is bounded on both l_1 and l_∞.

The following result extends lemma 1 of [4].

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LEMMA 1. *Let $\{e_n\}$ be a basis for a Banach space E with $\|e_n\|=1$. Let $x_n = \sum_{|j-n|\leqq k} a_{nj}e_j$ with $|a_{nj}|\leqq 1$ and $\inf_n \|x_n\|>0$. Then, $\{x_n\}$ is a basis for* span$\{x_n\}$ *if and only if there exist functionals $\{f_n\}\subseteq E^*$ such that $\{x_n, f_n\}$ form a bounded biorthogonal system.*

PROOF. Let $P_n(\sum_{i=1}^{\infty} t_i e_i) = \sum_{i=1}^{n} t_i e_i$ so $\sup_n \|P_n\| = K < \infty$. Now, for $n > m > k$,

$$\sum_{i=1}^{m} t_i x_i = \sum_{i=1}^{m} t_i \Big(\sum_{|j-i|\leqq k} a_{ij}e_j \Big) = \sum_{j=1}^{m-k} \Big(\sum_{|i-j|\leqq k} t_i a_{ij} \Big) e_j + \sum_{j=m-k+1}^{m+k} \Big(\sum_{\substack{|i-j|\leqq k \\ i\leqq m}} t_i a_{ij} \Big) e_j$$

$$= P_{m-k}\Big(\sum_{j=1}^{n} t_i x_i \Big) + \sum_{j=m-k+1}^{m+k} \Big(\sum_{\substack{|i-j|\leqq k \\ i\leqq m}} f_i(x) a_{ij} \Big) e_j$$

where $x = \sum_{i=1}^{n} t_i x_i$. This implies that for $n > m > k$,

$$\Big\| \sum_{i=1}^{m} t_i x \Big\| \leqq \Big(K + (2k+1)^2 \sup_i \|f_i\| \Big) \Big\| \sum_{i=1}^{n} t_i x_i \Big\|,$$

which shows that $\{x_i\}$ is a basis. The converse is well known.

The next lemma, which is the key to the theorem, is a slight extension of the main result of [2]; to make this paper self-contained, we include a proof.

LEMMA 2. *Let $A = (a_{ij})$ be a matrix, finite or infinite, satisfying*

(a) $|a_{ij}|\leqq 1$ *and there is an m such that $a_{ij}=0$ for $|i-j|>m$,*

(b) *there is a matrix $B=(b_{ij})$ such that $AB = I = BA$ and* $\sup\{\|Bx\|_p : \|x\|_1 = 1\} = M < \infty$ *for some $1\leqq p<\infty$.*

Then, there are constants $K>0$, $0<r<1$ depending on only M, m, and p such that $|b_{ij}|\leqq Kr^{|i-j|}$.

PROOF. Fix j and let $x = \sum_i b_{ij}e_i$ be the jth column of B. Let $x^{(k)} = \sum_{i\geqq j+k} b_{ij}e_i$. With $Ax^{(k)} = z^{(k)} = \sum_i z_i^{(k)} e_i$, we have by (a) and (b) that $z_i^{(k)} = 0$ if $i < j+k-m$ and $z_i^{(k-2m)} = 0$ if $i \geqq k+j-m$. Now,

$$\|x^{(k)}\|_p = \|BAx^{(k)}\|_p \leqq M\|z^{(k)}\|_1 \leqq M\|z^{(k)} - z^{(k-2m)}\|_1$$

$$\leqq (2m+1)M\|x^{(k)} - x^{(k-2m)}\|_1 \quad \text{since} \quad |a_{ij}|\leqq 1.$$

Consequently, $\sum_{i\geqq j+k} |b_{ij}|^p \leqq K \sum_{i=j+k-2m}^{j+k-1} |b_{ij}|^p$ where $K \leqq M^p(2m+1)^{2p-1}$. Now let $c_k = \sum_{i=j+k}^{j+k+2m-1} |b_{ij}|^p$ and $s_k = \sum_{i\geqq k} c_i$ so that $s_k \leqq 2mKc_{k-2m} = 2mK(s_{k-2m} - s_{k-2m+1})$. Therefore, $s_k + 2mKs_k \leqq s_k + 2mKs_{k-2m+1} \leqq 2mKs_{k-2m}$

and $s_k \leqq (2mK/(1+2mK))s_{k-2m}$. Consequently, for $0 \leqq t < 2m$ and $r \geqq 1$, $s_{2mr+t} \leqq (2mK/(1+2mK))^r s_0 \leqq (2mK/(1+2mK))^r 2mM$. This establishes the lemma for $i > j$. The same method works for $i < j$.

An immediate consequence of this lemma is that if A satisfies the above hypotheses, then A is an isomorphism on every interpolation space of l_1 and l_∞ since both A and A^{-1} are both bounded on l_1 and l_∞, cf. [5, pp. 219 ff].

THEOREM. *Let $\{x_n\}$, $x_n = \sum_{|j-n|\leqq m} a_{ij}e_j$, be a normalized banded perturbation of the unit vector basis $\{e_n\}$. Assume that there exist biorthogonal functionals $\{f_n\}$ with $\sup_n \|f_n\|_p = K < \infty$ for some $1 \leqq p < \infty$. Then, in every interpolation space of l_1 and l_∞, $\{x_n\}$ is equivalent to $\{e_n\}$.*

PROOF. We consider $\{x_n\}$ as a subset of l_q, $q^{-1} + p^{-1} = 1$. The codimension of $\operatorname{span}\{x_n\}$ is no greater than m (in the case $q = \infty$, we consider the codimension in c_0). If this codimension is k, let $e_{i_1}, \cdots, e_{i_k}$ be unit vector basis elements not in $\operatorname{span}\{x_n\}$ such that $\{e_{i_1}, \cdots, e_{i_k}\} \cup \{x_n\}$ is a basis for l_q. Let $\{y_n\} = \{x_n\} \cup \{e_{i_1}, \cdots, e_{i_k}\}$ be ordered so that $y_{i_j} = e_{i_j}$, $1 \leqq i \leqq k$, and so that if $y_i = x_r$, $y_j = x_s$ with $r < s$, then $i < j$. With this ordering $\{y_n\}$ is a normalized banded perturbation of $\{e_n\}$ of bandwidth at most $m + k$; and $\{y_n\}$ is a basis for l_q (c_0 in case $q = \infty$). Let $\{g_n\}$ be the associated sequence of uniformly bounded biorthogonal functionals in l_p. Let A be the infinite matrix whose nth row is the sequence y_n and let G be the matrix whose nth column is g_n. Note that G is a bounded operator from l_1 to l_p, [5, p. 220]. By biorthogonality $AG = I$. Now, for each n and k

$$\langle e_k, e_n\rangle = \sum_i g_i(e_n)\langle e_k, x_i\rangle = \sum_i g_i(e_n) \sum_{|j-i|\leqq m} a_{ij}\langle e_k, e_j\rangle$$

$$= \sum_{|i-k|\leqq m} a_{ik} g_i(e_n).$$

Therefore, $\sum_{|i-j|\leqq m} a_{ij} g_i(e_n) = \delta_{jn}$ and $GA = I$. This shows that $\{y_n\}$ is equivalent to $\{e_n\}$, but since $\{y_n\}$ and $\{x_n\}$ have the same "tails" we must have $\{x_n\}$ equivalent to $\{e_n\}$.

COROLLARY. *Let $\{x_n\}$ be a normalized banded perturbation of the unit vector basis in a reflexive Orlicz space l_M. Assume there exist uniformly bounded biorthogonal functionals $\{f_n\}$ in the dual space l_N. Then, $\{x_n\}$ is equivalent to $\{e_n\}$.*

PROOF. Since $\inf\{q : \inf_{0<x,\lambda\leqq 1}(N(\lambda x)/N(\lambda)x^q) > 0\} < \infty$, cf. [3], it follows that there is $M > 0$ and $q < \infty$ such that $\|f_n\|_q^q \leqq N(1)M^{-1}\|f_n\|_N^q$ for all n. Now apply the previous result.

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